Path Integral methods for solving stochastic problems

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Why?

- Often in neuroscience we run into stochastic ODEs of the form

$$\frac{dx}{dt} = f(x) + g(x)\eta(t)$$

$$\langle \eta(t) \rangle = 0 \qquad \langle \eta(t)\eta(t') \rangle = \delta(t - t')$$

Examples

- Integrate-and-fire neuron with noise

$$\frac{dv}{dt} = I - g_L v - g_{NL}(v) + \sqrt{D}\eta(t)$$

Set $v \to v_R$ when $v = v_T$

- Want to know mean and variance, mean and variance of time to fire, etc.

- "Decision-theory" model

$$\frac{du}{dt} = f(u) + g(t) + \sigma\eta(t)$$

$$u(0) = b$$

- Decision is made when u reaches $\pm\theta$

- Population of neurons with noise

$$\partial_t a = -a + f\left(\int w(x,y)a(y)dy\right) + \eta(x,t)$$

$$\langle \eta(x,t) \rangle = 0 \qquad \langle \eta(x,t)\eta(x',t') \rangle = \delta(x-x')\delta(t-t')$$

- Again interested in mean, variance, correlations, any statistic

Solving equations

- Traditional methods include Ito and Stratonovich calculus or Fokker-Planck equation
- For most nonlinear stochastic equations, closed form solutions do not exist
- Need to be solved with perturbation theory, which is nontrivial
- Path integral and field theory approaches were designed for perturbation theory

Generating Function

- Generating function lets you calculate moments of a probability density P(x) by taking derivatives

moments $$\langle x^n \rangle = \int x^n P(x) dx$$

generating function $$Z[\lambda] = \langle e^{\lambda x} \rangle = \int e^{\lambda x} P(x) dx$$

$$\langle x^n \rangle = \frac{1}{Z[0]} \frac{\partial^n}{\partial \lambda^n} Z[\lambda] \Big|_{\lambda=0}$$

Goals

- Derive a generating function for moments of a stochastic process
- Solve for moments perturbatively using diagrammatic methods.

Computing Generating functions

- e.g. Normal or Gaussian distribution

$$P(x) \propto e^{-\frac{(x-a)^2}{2\sigma^2}}$$

$$Z[\lambda] = \int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{2\sigma^2}+\lambda x} dx$$

Most important trick: complete the square

$$-\frac{(x-a)^2}{2\sigma^2}+\lambda x=-A(x-x_c)^2+B$$

x_c is given by $$\frac{d}{dx}\left(-\frac{(x-a)^2}{2\sigma^2}+\lambda x\right)=0$$

$$x_c=\lambda\sigma^2+a$$

$$A=\frac{1}{2\sigma^2}\qquad B=\frac{x_c^2}{2\sigma^2}-\frac{a^2}{2\sigma^2}=\frac{\lambda^2\sigma^2}{2}+\lambda a$$

$$Z[\lambda]=\int_{-\infty}^{\infty}e^{-\frac{(x-\lambda\sigma^2-a)^2}{2\sigma^2}+\lambda a+\frac{\lambda^2\sigma^2}{2}}dx=Z[0]e^{\lambda a+\frac{\lambda^2\sigma^2}{2}}$$

$$Z[0]=\sqrt{2\pi}\sigma$$

$$\langle x \rangle = \frac{d}{d\lambda}\, e^{\lambda a + \frac{\lambda^2\sigma^2}{2}}\Big|_{\lambda=0} = a$$

$$\langle x^2 \rangle = \frac{d^2}{d\lambda^2}\, e^{\lambda a + \frac{\lambda^2\sigma^2}{2}}\Big|_{\lambda=0} = a^2 + \sigma^2$$

$$\langle x^3 \rangle = \frac{d^3}{d\lambda^3}\, e^{\lambda a + \frac{\lambda^2\sigma^2}{2}}\Big|_{\lambda=0} = a^3 + 3a\sigma^2$$

$$\langle x^4 \rangle = \frac{d^4}{d\lambda^4}\, e^{\lambda a + \frac{\lambda^2\sigma^2}{2}}\Big|_{\lambda=0} = a^4 + 6a^2\sigma^2 + 3\sigma^4$$

Cumulant generating function

$$W[\lambda] = \ln Z[\lambda] \qquad C_n = \frac{d^n}{d\lambda^n} W[\lambda]\bigg|_{\lambda=0}$$

For Gaussian $\quad W[\lambda] = \lambda a + \frac{1}{2}(\lambda^2\sigma^2) + \ln Z[0]$

$$C_1 = \langle x \rangle = a$$

$$C_2 = \langle x^2 \rangle - \langle x \rangle^2 \equiv \mathrm{var}(x) = \sigma^2$$

$$C_n = 0, \quad n > 2$$

Generalize to multi-dimensions

$$Z[\lambda_i] = \int \prod_i dx_i e^{-\frac{1}{2}\sum_{j,k} x_j K^{-1}_{jk} x_k + \sum_j \lambda_j x_j} \qquad K^{-1}_{jk} \equiv (K^{-1})_{jk}$$

Assume K is symmetric and positive, thus it has full set of orthonormal eigenvectors

$$\sum_j K_{ij} v^\alpha_j = \omega_\alpha v^\alpha_i \qquad \sum_j v^\alpha_j v^\beta_j = \delta_{\alpha\beta}$$

Expand $$x_k = \sum_\alpha c_\alpha v^\alpha_k \qquad \lambda_k = \sum_\alpha d_\alpha v^\alpha_k$$

$$\sum_{j,k} x_j K_{jk} x_k = \sum_j \sum_{\alpha,\beta} c_\alpha \omega_\beta c_\beta v^\alpha_j v^\beta_k = \sum_{\alpha,\beta} c_\alpha \omega_\beta c_\beta \delta_{\alpha\beta} = \sum_\alpha \omega_\alpha c^2_\alpha$$

$$Z[\lambda_i] = \int \prod_i dx_i e^{-\frac{1}{2}\sum_{j,k} x_j K^{-1}_{jk} x_k + \sum_j \lambda_j x_j}$$

$$= \int \prod_\alpha dc_\alpha e^{\sum_\alpha (-\frac{1}{2}\omega_\alpha c_\alpha^2 + d_\alpha c_\alpha)}$$

$$= \prod_\alpha \int_{-\infty}^{\infty} dc_\alpha e^{-\frac{1}{2}\omega_\alpha c_\alpha^2 + d_\alpha c_\alpha}$$

$$= Z[0] \prod_\alpha e^{\frac{1}{2}\omega_\alpha^{-1} d_\alpha^2} \qquad Z[0] = (2\pi \det K)^{n/2}$$

$$= Z[0] e^{\sum_{jk} \frac{1}{2}\lambda_j K_{jk} \lambda_k} \qquad \sum_j K^{-1}_{ij} K_{jk} = \delta_{ik}$$

Cumulant generating function $W[\lambda_i] = \ln Z[\lambda_i]$

Moments

$$\langle \prod_i x_i \rangle = \frac{1}{Z[0]} \prod_i \frac{\partial}{\partial \lambda_i} Z[\lambda_i] \Big|_{\lambda_i = 0}$$

Example

$$\langle x_3^2 x_7 x_{36} \rangle = \frac{1}{Z[0]} \frac{\partial^2}{\partial \lambda_3^2} \frac{\partial}{\partial \lambda_7} \frac{\partial}{\partial \lambda_{36}} \exp \left(\sum_{ij} \frac{1}{2} \lambda_i K_{ij}^{-1} \lambda_j \right) \Big|_{\lambda_i = 0}$$

$$= K_{3,3} K_{7,36} + 2 K_{3,7} K_{3,36}$$

Wick's Theorem

$$\langle \prod_1^{2s} x_i \rangle = \frac{1}{Z[0]} \prod_1^{2s} \frac{\partial}{\partial \lambda_i} Z[\lambda_i] \Big|_{\lambda_i = 0}$$

$$= \sum_{\text{all possible pairings}} K_{i_1, i_2} \cdots K_{i_{2s-1} i_{2s}}$$

$$\langle x_a x_b x_c x_d \rangle = \overline{x_a x_b}\,\overline{x_c x_d} + \overline{x_a \overline{x_b x_c} x_d} + x_a x_b x_c x_d$$

$$= K_{ab} K_{cd} + K_{ad} K_{bc} + K_{ac} K_{bd}$$

make all pairings or "contractions"

Continuum limit

$$Z[\lambda_i] = \int \prod_i dx_i e^{-\frac{1}{2}\sum_{j,k} x_j K^{-1}_{jk} x_k + \sum_j \lambda_j x_j}$$

Let $i \to t \quad x_i \to x(t) \quad \lambda_i \to \lambda(t) \quad K_{ij} \to K(s,t)$

$$\sum_i \to \int dt \qquad \prod_i dx_i \to \mathcal{D}x(t)$$

$$Z[\lambda(t)] = \int \mathcal{D}x(t) e^{-\frac{1}{2}\int x(s)K^{-1}(s,t)x(t)dsdt + \int \lambda(t)x(t)dt}$$

$$= Z[0]e^{\int \frac{1}{2}\lambda(s)K(s,t)\lambda(t)dsdt}$$

functional derivative

$$\frac{\partial}{\partial \lambda_i} \to \frac{\delta}{\delta \lambda(t)} \qquad \frac{\delta \lambda(s)}{\delta \lambda(t)} = \delta(s-t)$$

Moments

$$\langle x(t_1)x(t_2)\rangle = \frac{1}{Z[0]}\frac{\delta}{\delta\lambda(t_1)}\frac{\delta}{\delta\lambda(t_2)}Z[\lambda(t)] \quad = K(t_1, t_2)$$

$$\langle \prod_i x(t_i)\rangle = \frac{1}{Z[0]}\prod_i \frac{\delta}{\delta\lambda(t_i)}Z[\lambda(t)]$$

$$= \sum_{\text{all possible contractions}} K(t_{i_1}, t_{i_2}) \cdots K(t_{i_{2s-1}}, t_{t_{i_{2s}}})$$

Can generalize to higher dimensions

$$x(t) \to \varphi(\vec{t}) \qquad \lambda(t) \to J(\vec{t}) \qquad \vec{t} \in \mathbb{R}^d$$

$$\int dt \to \int d^d t$$

$$Z[\lambda(\vec{t})] = \int \mathcal{D}\varphi e^{-\frac{1}{2}\int \varphi(\vec{s})K^{-1}(\vec{s},\vec{t})\varphi(\vec{t})d^d s d^d t + \int \lambda(\vec{t})\varphi(\vec{t})d^d t}$$

$$= Z[0]e^{\int \frac{1}{2}\lambda(\vec{s})K(\vec{s},\vec{t})\lambda(\vec{t})d^d s d^d t}$$

Field Theory

Consider an arbitrary (infinite dimensional) probability distribution

$$P[\varphi(\vec{t})] = e^{-S[\varphi(\vec{t})]}$$

S is called the "action"

$$Z[J(\varphi(\vec{t})] = \int \mathcal{D}\varphi e^{-S[\phi]+J\cdot\varphi}$$

$$J\cdot\varphi = \int J(\vec{t})\varphi(\vec{t})d^dt$$

e.g. φ^4 theory $S[\varphi(\vec{t})] = \int \varphi(\vec{t})K^{-1}(\vec{t},\vec{t'})\varphi(\vec{t'})d^dtd^dt' + g\int \varphi^4(\vec{t})d^dt$

Must compute perturbatively (asymptotic solutions of integrals in infinite dimensions)

Application to stochastic differential equations

$$\frac{dx}{dt} = f(x) + g(x)\eta(t)$$

Ito interpretation

$$dx = f(x)dt + g(x)dB_t$$

Discretization

$$x_{i+1} - x_i = f(x_i)dt + g(x_i)w_i\sqrt{dt} \qquad dB_t = w_t\sqrt{t}$$

Probability density

$$P[x_i|w_i] = \delta[x_{i+1} - x_i - f(x_i)dt - g(x_i)w_i\sqrt{dt}]$$

$$= \cdots \delta(x_1 - x_0 - f(x_0)dt - g(x_0)w_0\sqrt{dt})\delta(x_2 - x_1 -$$

Use Fourier transform representation

$$\delta(z) = \frac{1}{2\pi}\int e^{-ik_i z} dk_i$$

$$P[x_i|w_i] = \int \prod_i \frac{dk_i}{2\pi} e^{-i\sum_i k_i(x_{i+1} - x_i - f(x_i)dt - g(x_i)w_i\sqrt{t})}$$

$$P[x_i|w_i] = \int \prod_i \frac{dk_i}{2\pi} e^{-i\sum_i k_i(x_{i+1}-x_i-f(x_i)dt)} e^{ik_i g(x_i) w_i \sqrt{dt}}$$

w_i is Normal or Gaussian N(0,1) with density

$$P[w_i] = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w_i^2}$$

$$P[x_i] = \int \prod_i dw_i P[x_i|w_i] P[w_i]$$

$$= \int \prod_i \frac{dk_i}{2\pi} e^{-i\sum_i k_i(x_{i+1}-x_i-f(x_i)dt)} \int \prod_i \frac{dw_i}{\sqrt{2\pi}} e^{ik_i g(x_i) w_i \sqrt{dt}} e^{-\frac{1}{2}w_i^2}$$

Complete the square

$$P[x_i] = \int \prod_i \frac{dk_i}{2\pi} e^{-\sum_i (ik_i)(\frac{x_{i+1}-x_i}{dt} - f(x_i))dt + \sum_i \frac{1}{2} g^2(x_i)(ik_i)^2 dt}$$

Make a change of variables $\tilde{k}_i = ik_i$

Take the continuum limit

$$P[x(t)] = \int \mathcal{D}\tilde{k} e^{-\int dt \tilde{k}(\dot{x} - f(x(t)) + \frac{1}{2}\tilde{k}^2 g^2(x(t))}$$

action $S[x(t), \tilde{k}(t)] = \int dt \tilde{k}(\dot{x} - f(x(t)) - \frac{1}{2}\tilde{k}^2 g^2(x(t))$

From SDE to action

$$\dot{x} = f(x) + g(x)\eta(t)$$

$$P[x] = \int \mathcal{D}\eta\delta[\dot{x} - f(x) - g(x)\eta(t)]e^{-\int \eta^2(t)dt}$$

$$P[x] = \int \mathcal{D}\eta\mathcal{D}\tilde{k}e^{-\int \tilde{k}(\dot{x}-f(x))+\tilde{k}g(x)\eta(t)-\eta^2(t)dt}$$

$$P[x] = \int \mathcal{D}\tilde{k}e^{-\int \tilde{k}(\dot{x}-f(x))+\frac{1}{2}\tilde{k}^2g^2(x)dt}$$

$$S[x(t), \tilde{k}(t)] = \int dt\tilde{k}(\dot{x} - f(x(t)) - \frac{1}{2}\tilde{k}^2g^2(x(t))$$

Ornstein-Uhlenbeck

$$\dot{x} = -ax + \sqrt{D}\eta(t)$$

$$P[x] = \int \mathcal{D}\tilde{k} e^{-\int \tilde{k}(\dot{x}+ax)+\frac{D}{2}\tilde{k}^2 dt} \qquad G^{-1}(t-t') = (\frac{d}{dt}+a)\delta(t-t')$$

$$P[x] = \int \mathcal{D}\tilde{k} e^{-\int dt dt' \tilde{k}(t) G^{-1}(t-t') x(t') + \int dt \frac{D}{2}\tilde{k}^2}$$

$$S[x,\tilde{k}] = \int dt dt' \tilde{k}(t) G^{-1}(t-t') x(t') - \int dt \frac{D}{2}\tilde{k}^2$$

Generating functional

$$Z[J(t),\tilde{J}(t)] = \int \mathcal{D}x\mathcal{D}\tilde{k}e^{-\int dtdt'\tilde{k}G^{-1}x+\int \frac{D}{2}\tilde{k}^2dt+\int \tilde{J}xdt+\int J\tilde{k}dt}$$

$$= \int \mathcal{D}x\mathcal{D}\tilde{k}e^{-\int dtdt'\tilde{k}G^{-1}x}(1+\mu+\frac{1}{2!}\mu^2+\frac{1}{3!}\mu^3+\cdots)$$

where $\mu = \int \frac{D}{2}\tilde{k}^2dt + \int \tilde{J}xdt + \int J\tilde{k}dt$

Expansion in moments of “complex” Gaussian

$$Z_{CG}[J(t),\tilde{J}(t)] = \int \mathcal{D}x\mathcal{D}k e^{-\int dt dt' ikG^{-1}x + \int ik(t)J(t)dt + \int x(t)\tilde{J}(t)dt}$$

Restore $\tilde{k} = ik$ for convenience

Discretize and diagonalize $\int dt' G^{-1}x \to \lambda(\omega)\hat{x}(\omega)$

Expand in $\hat{x}(\omega)$ $k, J, \tilde{J} \to \hat{k}(\omega), \hat{J}(\omega), \hat{\tilde{J}}(\omega)$

$$Z_{CG}[\hat{J},\hat{\tilde{J}}] = \int \mathcal{D}\hat{x}\mathcal{D}\hat{k} e^{-\sum_\omega \; i\hat{k}\hat{x}\lambda + i\hat{k}\hat{J} + \hat{x}\hat{\tilde{J}}}$$

$$= \prod_\omega \int \frac{d\hat{x}d\hat{k}}{2\pi} e^{-i\hat{k}\hat{x}\lambda + i\hat{k}\hat{J} + \hat{x}\hat{\tilde{J}}}$$

$$= \prod_{\omega} \int \frac{d\hat{x} d\hat{k}}{2\pi} e^{-i\hat{k}(\hat{x}\lambda - \hat{J}) + \hat{x}\hat{\tilde{J}}}$$

$$= \prod_{\omega} \int d\hat{x} \delta(\hat{x}\lambda - \hat{J}) e^{\hat{x}\hat{\tilde{J}}}$$

$$= \prod_{\omega} e^{\lambda^{-1}\hat{j}\hat{\tilde{J}}}$$

Transform back to get

$$Z_{CG}[J, \tilde{J}] = e^{\int J(t) G(t,t') \tilde{J}(t')}$$

Propagator

$$(\frac{d}{dt}+a)x = \int G^{-1}(t,t')x(t')dt'$$

$$G^{-1}(t,t') = (\frac{d}{dt}+a)\delta(t-t')$$

$$\int G^{-1}(t,t')G(t',t'')dt' = \delta(t-t'')$$

$$(\frac{d}{dt}+a)G(t_1,t_2) = \delta(t_1-t_2)$$ Green's function

Moments

$$\langle \prod_{ij} x(t_i)\tilde{k}(t_j)\rangle = \prod_{ij} \frac{\delta}{\delta \tilde{J}(t_i)} \frac{\delta}{\delta J(t_j)} e^{\int \tilde{J}(t)G(t,t')J(t')dtdt'}\Big|_{J=\tilde{J}=0}$$

Only surviving moments have equal numbers of x and $\tilde{k}$

Wick's theorem still applies but only for contractions between x and $\tilde{k}$

e.g. $\langle x(t_1)x(t_2)\tilde{k}(t_3)\tilde{k}(t_4)\rangle = G(t_1,t_3)G(t_2,t_4) + G(t_1,t_4)G(t_2,t_3)$

Back to the OU problem

$$Z[J(t), \tilde{J}(t)] = \int \mathcal{D}x\mathcal{D}\tilde{k} e^{-\int dt dt' \tilde{k} G^{-1} x + \int \frac{D}{2} \tilde{k}^2 dt + \int \tilde{J} x dt + \int J \tilde{k} dt}$$

$$= \int \mathcal{D}x\mathcal{D}\tilde{k} e^{-\int dt dt' \tilde{k} G^{-1} x} (1 + \mu + \frac{1}{2!}\mu^2 + \frac{1}{3!}\mu^3 + \cdots)$$

where $\mu = \int \frac{D}{2} \tilde{k}^2 dt + \int \tilde{J} x dt + \int J \tilde{k} dt$

$$\equiv \mu_1 + \mu_2 + \mu_3$$

Can see that only some combinations of the μ terms will survive i.e. $\mu_2\mu_3$ and $\mu_1\mu_2^2$

$$Z[J(t),\tilde{J}(t)] = \int \langle \tilde{J}(t)J(t')x(t)\tilde{k}(t')\rangle dt dt' + \frac{D}{4}\int \langle \tilde{J}(t)\tilde{J}(t')x(t)x(t')\tilde{k}^2(t'')\rangle dt dt' dt'' + \ldots$$

$$\langle x(t_1)x(t_2)\rangle = \frac{D}{2}\int \langle x(t_1)x(t_2)\tilde{k}^2(t'')\rangle dt''$$

$$= D\int G(t_1,t'')G(t_2,t'')dt''$$

Feynman diagrams

Each term in action can be represented by a "diagram"

Turn perturbation theory into rules for assembling diagrams

Diagrams consist of edges and nodes, representing terms in the action

Every factor of $\tilde{k}$ is given an outgoing edge

Every factor of x is given an incoming edge

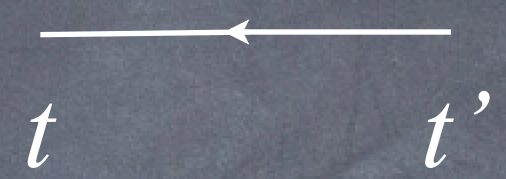
Convention is time goes from right to left

Propagators are lines, other terms are vertices

$$S[x,\tilde{k}] = \boxed{\int dt dt' \tilde{k}(t) G^{-1}(t,t') x(t')} - \boxed{\int dt \frac{D}{2} \tilde{k}^2}$$

Assign a diagram to each term in the action

Propagator $G(t,t')$



Other diagrams are called vertices

Vertex $\frac{D}{2}$

sign negative from action

Feynman rules

Surviving terms in perturbation series of generating functional consist of equal numbers of $\tilde{k}$ and x factors

For vertices, this means that outgoing edges are joined to incoming edges*

Propagators are then attached to all edges in all possible ways (accounts for Wick's theorem)*

Vertices are integrated over

Moment rules

Moments of x are given by derivatives over $\tilde{J}$, moments of $\tilde{k}$ are given by derivatives over J

Thus moments can also be expressed as diagrams

Each factor of x is given an outgoing edge and each factor of $\tilde{k}$ is given an incoming edge

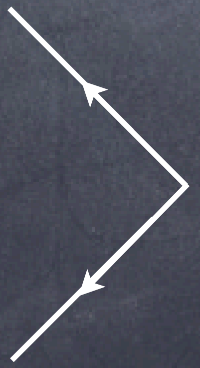
Cumulants are connected diagrams

Examples

Propagator:

$$\langle x(t_1)\tilde{k}(t_2)\rangle \quad = \quad$$  $$\quad = \quad G(t_t, t_2)$$

2nd moment:

$$\langle x(t_1)x(t_2)\rangle \quad = \quad$$  $$= 2 \times \int G(t_1, t')G(t_2, t')\frac{D}{2}dt'$$

Doing the integrals

$$(\frac{d}{dt} + a)G(t_1, t_2) = \delta(t_1 - t_2)$$

$$G(t_1, t_2) = e^{-a(t_1 - t_2)} H(t_1 - t_2) \qquad H(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

$$D \int G(t_1, t')G(t_2, t')dt' = D \int_0^{t_2} e^{-a(t_1 - t')} e^{-a(t_2 - t')} dt' \qquad t_2 > t_1$$

$$= D \frac{e^{2a(t_1 - t_2)} - e^{-2a(t_1 + t_2)}}{2a}$$

$$= \frac{D}{2a}(1 - e^{-2at}) \qquad t_1 = t_2 = t$$

Nonlinear terms

$$\dot{x} = -ax + bx^2 - cx^3 + \sqrt{A + Bx + Cx^2}\eta(t) + x_0\delta(t - t_0)$$

Using the action formula

$$S[x(t), \tilde{k}(t)] = \int dt\tilde{k}(\dot{x} - f(x(t)) - \frac{1}{2}\tilde{k}^2 g^2(x(t))$$

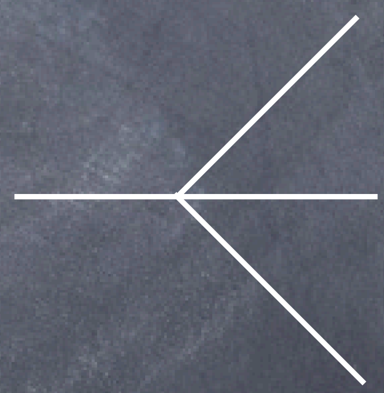
$$S[x(t), \tilde{k}(t)] = \int dt\tilde{k}(\dot{x} + ax - bx^2 + cx^3 - x_0\delta(t - t_0)) - \tilde{k}^2\frac{A + Bx + Cx^2}{2}$$

$$S[x(t),\tilde{k}(t)] = \int dt\tilde{k}(\dot{x} + ax - bx^2 + cx^3 - x_0\delta(t-t_0)) - \tilde{k}^2\frac{A + Bx + Cx^2}{2}$$

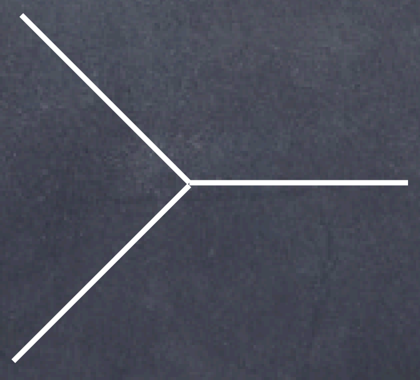
Propagator  $G(t,t')$

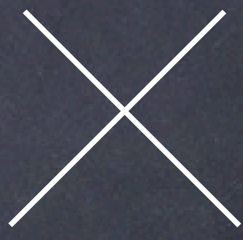
Vertices $x_0\delta(t-t_0)$ 

b 

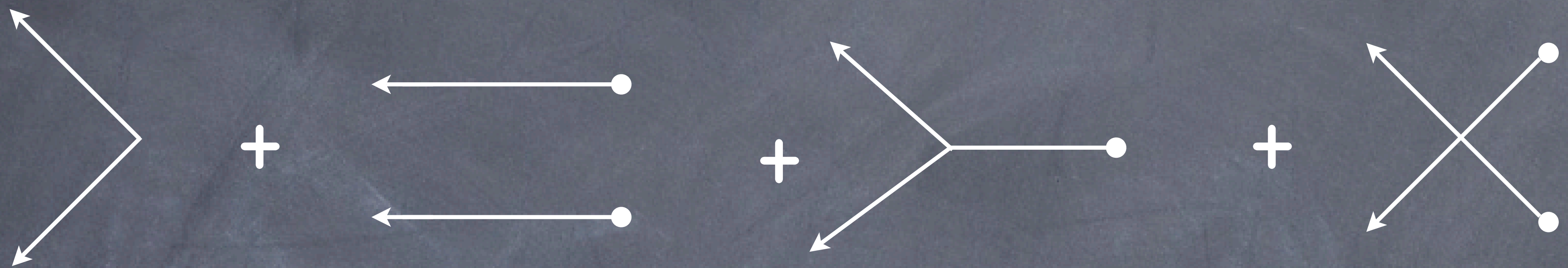
c 

$\frac{A}{2}$ 

$\frac{B}{2}$ 

$\frac{C}{2}$ 

$$\langle x(t_1)x(t_2)\rangle =$$



$$2\frac{B}{2}x_0\int G(t_1,t')G(t_2,t')G(t',t_0)dt'$$

Tree level

One loop

Acknowledgments and References

I will be writing a review that will be posted on my blog at sciencehouse.wordpress.com

- Michael Buice
- Zinn-Justin, Quantum Field Theory and Critical Phenomena
- Kardar, Statistical Physics of Fields